Name:

Student ID:

**Math 3083 PRACTICE Midterm Exam**

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The exam contains five problems which are worth 20 points each. The extra credit problem is worth 20 points additional points. You may use your book and any notes you may have. A statement is considered TRUE in Problems 1 and 2 if it is necessarily true, that is, if there are no counterexamples. If you have any questions about the meaning of any of the words or notation on the test, please ask.

| Prob 1 | Prob 2 | Prob 3 | Prob 4 | Prob 5 | ExCred | Total Score |
|---|---|---|---|---|---|---|
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**Problem 1. (TRUE/FALSE)**
Circle the letter corresponding to the best answer.

Let $A$, $B$, and $C$ be sets. Then $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.

**(T)** True.

**(F)** False.

Let $A$ and $B$ be sets and let $f : A \to B$ be an injective function. Then $|A| \leq |B|$.

**(T)** True.

**(F)** False.

Every nonempty subset of $\mathbb{N}$ contains a minimal element.

**(T)** True.

**(F)** False.

Let $C_1$ and $C_2$ be Dedekind cuts, and set $C = \{c_1 - c_2 \mid c_1 \in C_1 \text{ and } c_2 \in C_2\}$. Then $C$ is a Dedekind cut.

**(T)** True.

**(F)** False.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{6n} = L$.
Then $\lim_{n\to\infty} a_{3n} = L$.

**(T)** True.

**(F)** False.

**Problem 2. (Multiple Choice)**
Circle the letter corresponding to the best answer.

Let $f : A \to B$ and $g : B \to A$ be functions. Suppose that $f$ is injective.

**(a)** If $g$ is injective, then $f \circ g = \mathrm{id}_B$.

**(b)** If $g$ is surjective, then $g \circ f = \mathrm{id}_A$.

**(c)** Both **(a)** and **(b)**.

**(d)** Neither **(a)** nor **(b)**.

Let $S \subset \mathbb{R}$ be a bounded set and let $T = \{x \in \mathbb{R} \mid x = -s \text{ for some } s \in S\}$.

**(a)** Then $\inf S \leq \sup T$.

**(b)** Then $\inf S \leq -\sup T$.

**(c)** Both **(a)** and **(b)**.

**(d)** Neither **(a)** nor **(b)**.

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and let $S = \{s_n \mid n \in \mathbb{Z}^+\}$.

**(a)** Then $\limsup s_n \leq \sup S$.

**(b)** Then $\liminf s_n \geq \inf S$.

**(c)** Both **(a)** and **(b)**.

**(d)** Neither **(a)** nor **(b)**.

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers.

**(a)** Then $\{s_n\}_{n=1}^{\infty}$ has a convergent subsequence.

**(b)** Then $\{s_n\}_{n=1}^{\infty}$ has a divergent subsequence.

**(c)** Both **(a)** and **(b)**.

**(d)** Neither **(a)** nor **(b)**.

Let $U$ be an open set and let $F$ be a closed set.

**(a)** Then $U \cup F$ is open.

**(b)** Then $U \cap F$ is closed.

**(c)** Both **(a)** and **(b)**.

**(d)** Neither **(a)** nor **(b)**.

**Problem 3. (Computation)**

**(a)** Let $r \in \mathbb{R}$. Use induction to show that $1 - r^{n+1} = (1-r)(1 + r + \cdots + r^n)$ for every $n \in \mathbb{Z}^+$.

**(b)** Let $r \in \mathbb{R}$ with $|r| < 1$. Let $s_n = \sum_{i=0}^{n} r^i$. Show that $\lim_{n\to\infty} s_n = \frac{1}{1-r}$.

**Problem 4. (Examples)**
**(a)** Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be bounded sequences of real numbers.
Show that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

**(b)** Find an example of a pair of bounded sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfying

$$\limsup(a_n + b_n) < \limsup a_n + \limsup b_n.$$

**Problem 5. (Theory)**
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be bounded sequences of positive real numbers, and suppose that $\{a_n+b_n\}_{n=1}^{\infty}$ has a subsequence which converges to $c \in \mathbb{R}$. Show that there exists a subsequence of $\{a_n\}_{n=1}^{\infty}$ which converges to $a \in \mathbb{R}$ and a subsequence of $\{b_n\}_{n=1}^{\infty}$ which converges to $b \in \mathbb{R}$ such that $c = a + b$.
(Hint: use the knowledge that every bounded sequence has a convergent subsequence.)

**Problem 6. (Extra Credit)**
Construct a divergent sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that the subsequence $\{a_{mk}\}_{k=1}^{\infty}$ converges for every positive integer $m$ with $m \geq 2$.